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NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.

It is proposed in these papers to give a more or less complete list of proofs, both new and old, of this celebrated and practical theorem. An attempt is made at classification based upon immediate principles used in the proofs. Due credit will be given in all known cases. A historical note will be appended to the completed list.

THEOREM.

The square described upon the hypotenuse of a right triangle is equivalent to the sum of the squares described upon the other two sides.

PROOFS.

I. RESULTING FROM LINEAR RELATIONS OF SIMILAR RIGHT TRIANGLES.

Let ABC be a $\triangle$ right-angled at C. Draw CD perpendicular to AB. There are thus three similar right triangles.

Letting $AC=b$, $BC=a$, $AB=c$, $CD=x$, $AD=y$, $BD=c-y$, we obtain the following proportions, with their resulting equations :

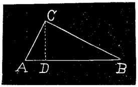


Fig. 1.

(1). $y : b :: b : c$. $\therefore yc=b^2$ ....1.

(2). $y : b :: x : a$. $\therefore bx=ay$ ....2.

(3). $b : c :: x : a$. $\therefore cx=ab$ ....3.

(4). $y : x :: x : c-y$. $\therefore x^2=cy-y^2$ 4

(5). $y : x :: b : a.$ $\therefore bx=ay$ .... 2.

(6). $x : c-y :: b : a.$ $\therefore ax=b(c-y)$ .... 5.

(7). $c-y : a :: a : c.$ $\therefore c(c-y)=a^2$ .... 6.

(8). $c-y : a :: x : b.$ $\therefore ax=b(c-y)$ .... 5.

(9). $a : c :: x : b.$ $\therefore cx=ab$ .... 3.

From the nine different proportions, there are derived but six different equations, equation 2 being derived from proportion (2) or (5), 3 from (3) or (9), and 5 from (6) or (8).

It is evident that from no single equation can we determine the relation between a, b, and c, the sides of the given right $\triangle$.

It is also evident that there is but one set of twos which will give the relation desired, viz., equations 1 and 6. If we add these, member by member, we get directly $c^2=a^2+b^2$. Giving to this form the usual geometrical interpretation, we thus have one proof of the theorem. This, though in a different form, is one of the methods usually found in the books. It is credited to Legendre.

We now proceed to find combinations of threes, which will give the required relation. There are $\frac{6.5.4}{\lfloor 3}=20$ sets of three equations out of the six. But of these, four must be rejected, since they contain 1 and 6, which two alone prove the theorem, as already shown; also the following three sets, since in each set the equations are dependent: 1, 2, 3; 2, 4, 5; 3, 5, 6. There are, then, left the following thirteen sets, from each of which, if we eliminate x and y, we get $c^2=a^2+b^2$: 1, 2, 4; 1, 2, 5; 1, 3, 4; 1, 3, 5; 1, 4, 5; 2, 3, 4; 2, 3, 5; 2, 3, 6; 2, 4, 6; 2, 5, 6; 3, 4, 5; 3, 4, 6; 4, 5, 6.

Of these thirteen sets, there are six that contain one equation each, derived from either of two proportions; six sets containing two each such equations; and one containing three. Therefore, including the proof already given, there are $1+6\times2+6\times2^2+2^3=45$ proofs, by this method.

II. Let ABC be a $\triangle$ right-angled at C. Draw a line perpendicular to AB from A, meeting BC produced, as at D.

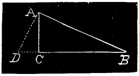


Fig. 2.

Letting $AC=b$, $BC=a$, $AB=c$, $AD=x$, $DC=y$, $BD=y+a$, and proceeding as in the preceding case, we find that this method also yields 45 different proofs. The details are left to be carried out by the reader.

III. Let ABC be a $\triangle$ right-angled at C. Draw DE perpendicular to AB so that $DE=DC$. Then will $BE=BC$. $\triangle ADE$ is similar to $\triangle ABC$.

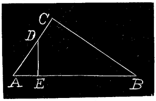


Fig. 3.

Letting $AC=b$, $BC=a$, $AB=c$, $AE=c-a$, $DE=DC=x$, $AD=b-x$, we have :

(1). $c-a : b :: x : a$. $\therefore x=\dfrac{ac-a^2}{b}$ .... 1.

(2). $c-a : b :: b-x : c$. $\therefore x=\dfrac{b^2-c^2+ac}{b}$ .... 2.

(3). $x : a :: b-x : c$. $\therefore x=\dfrac{ab}{a+c}$ .... 3.

From the three equations, it is evident that we may obtain three proofs by this method.

IV. Let ABC be a $\triangle$ right-angled at C. Extend AB to D making $BD=BC$. Draw a line perpendicular to AD at D, meeting AC produced as at E. Then will $CE=DE$, and $\triangle$ AED will be similar to $\triangle$ ABC.

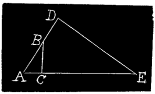


Fig. 4.

Letting $AC=b$, $BC=a$, $AB=c$, $AD=c+a$, $DE=x$, $AE=x+b$, and proceeding as in the last case, we obtain three more proofs, making in all, thus far, 96 proofs.

In our next paper, we shall give a method whose results reach into the thousands.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from February Number.]

PROPOSITION XXIII. *If any two straights AX, BX (Fig. 27.) exist in the same plane, either they have (even in the hypothesis of acute angle) a common perpendicular, or prolonged toward either the same part, unless sometime at a finite distance one strikes upon the other, they mutually approach ever more toward each other.*

Proof. From any point A of AX is let fall to the straight BX the perpendicular AB. If BA makes with AX a right angle, we have the asserted case of the common perpendicular. But otherwise this straight makes toward one or the other part, as suppose toward the parts of the point X, an acute angle.

So in the aforesaid straight AX between the points A and X any points D, H, L are designated, from which are let fall to the straight BX the perpendiculars DK, HK, LK. If any one angle at the points D, H, L be acute toward the parts of the point A, it follows (from the preceding) that AX, BX will have a common perpendicular.

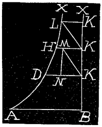


Fig. 27.

But if every angle of this sort be greater than acute; either some one will be right, and thus again we will have the asserted case of the common perpendicular, since all angles at the points K are supposed right; or all those angles toward the parts of the point A are obtuse, and therefore all therewith acute toward the parts of the point X, and so again I argue: Since in the quadrilateral $KDHK$ the angles at the points K are right, but the angle at the point D is acute, the side DK will be (from Cor. II. after P. III.) greater than the side HK.

In a similar way the side HK is shown to be greater than the side LK; and so always, comparing to each other perpendiculars from any ever higher points of AX let fall upon the other BX. Wherefore AX, BX mutually approach each other ever more toward the parts of the point X: Which is the second part of the disjunct proposition.

From all which follows that any two straights AX, BX, which exist in the same plane, either have (even in the hypothesis of acute angle) a common perpendicular, or produced toward either the same part, unless sometime at a finite distance one strikes upon the other, mutually approach each other ever more.

Quod erat etc.

Corollary I. Hence the angles toward the base AB will be always obtuse at each point of AX, from which is let fall a perpendicular to the straight BX: will be, I say, always obtuse, as often as those two AX and BX mutually approach each other ever more toward the parts of the points X; which indeed ought to be understood in a sane way, of course, of perpendiculars let fall before the aforesaid meeting, if perchance one should strike upon the other at a finite distance.

Scholion. I see indeed that it may be here inquired, in what way can be shown the existence of that common perpendicular, as often as any straight $PFHD$ (Fig. 28.) meeting two AX, BX in points F, and H, makes toward the same parts two internal angles AHF, BFH, not themselves indeed right, but nevertheless together equal to two rights. But behold that common perpendicular geometrically demonstrated.

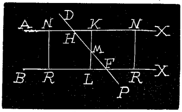


Fig. 28.

FH being bisected in M, perpendiculars MK, ML are let fall to AX and BX. The angle MFL will be equal (Eu. I. 13) to the angls MHK, which indeed is supposed to make up two right angles with the angle BFH. Moreover the angles at the points K and L are

right ; and again MF, MH are equal. Therefore (Eu. I. 26) so are the angles FML, HMK equal. Wherefore the angle HMK makes two right angles with the angle HML, since with this the angle FML (Eu. I. 13) makes two right angles. Therefore (Eu. I. 14) KML will be one continuous straight line, consequently the common perpendicular of the aforesaid straights AX, BX. Quod erat etc.

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from February Number.]

The Construction of the Primitive Groups.

We have shown that all the intransitive and the non-primitive groups of a given degree, may be made to depend upon groups of a lower degree. We shall soon prove a similar property of the primitive groups.

It must however not be inferred that this will solve, in a satisfactory manner, the problem of constructing all the groups of a given degree. The elementary methods to which we have confined ourselves require a large number of trials if the degree is large. Some briefer methods will be given later but even these will only tend to make the construction of all the groups of a given degree practical for somewhat larger degrees.

It is not difficult to give general theorems which include all the groups of a given type, as, for instance, the theorem at the end of our discussion of the construction of the non-primitive groups ; but new types arise continually and no non-tentative method by means of which all the groups of any given degree may be found has yet been published.

We proceed to prove some theorems which apply to all transitive groups but are especially useful in the construction of primitive groups. Unless the contrary is stated the symbols G, g, and n will respresent respectively the group under consideration, its order and its degree.

Let us consider the transitive group G which contains the letters $a_1, a_2, \ldots\ldots, a_n$. The substitutions of G which do not contain a_1 (*i. e.*, those which replace a_1 by itself) may be represented by

$$s_1,\ s_2,\ \ldots\ldots,\ s_r \equiv G_1.$$

As every group must contain the identical substitution if the number of its

letters is finite and this is the only kind we are considering now, the minimum value of r is unity.

Since G is transitive it must contain some substitution s_{r+1} which replaces a_1 by a_2. We desire to find all the substitutions of G which have this property. If s_k is such a substitution then will

$$s_k s_{r+1}^{-1}$$

belong to the first line, since s_k replaces a_1 by a_2 and s_{r+1}^{-1} replaces a_2 by a_1 $s_k s_{r+1}^{-1}$ must leave a_1 unchanged. Hence we have the equations

$$s_k s_{r+1}^{-1} = s_\alpha \qquad (\alpha = 1, 2, \ldots\ldots r)$$

$$s_k = s_\alpha \; s_{r+1}.$$

Since the condition expressed by the last equation is sufficient as well as necessary it follows that there are just r different substitutions in G, which transform a_1 into a_2. Similarly there are exactly r substitutions in G which replace a_1 by a_3, etc. From this we see that the number of substitutions which replace a_1 by itself is equal to the number of those which replace a_1 by any other letter of G. We have imposed no condition upon a_1 which is not satisfied by each of the other letters so that the property which we have proved in regard to a_1 belongs to all the letters. That r has the same value for each of the letters follows from the following considerations :

If the substitutions of G which do not contain a_α are

$$s_1^1, \; s_2^1 \ldots\ldots s_{r^1}^1 \equiv G_2,$$

then will

$$s_{r+1} G_2 s_{r+1}^{-1} = r^1 \text{ substitutions of } G \text{ which do not contain } a_1 \text{ and}$$

$$s_{r+1}^{-1} G_1 s_{r+1} = r \text{ substitutions of } G \text{ which do not contain } a_2.$$

From the first of these two equations we have $r' \overline{\gtrless} r$ and from the second $r \overline{\gtrless} r'$, hence $r' = r$. Similar remarks clearly apply to all the letters of G. We may embody the results at which we have arrived in the following

THEOREM: *The number of substitutions (r) of any transitive group (G), which do not contain any given letter, is equal to the number of substitutions which replace a letter by any required other letter of the group.*

Corollary I. $g = nr$, *i. e. the order of any transitive group is a multiple of its degree.*

*Corollary II. The average number of letters in all the substitutions of a transitive group of degree n is $n-1$.**

*Since every intransitive group may be resolved into transitive constituent groups whose separate elements enter an equal number of the substitutions of the intransitive group, the general statement of this corollary is as follows: *The average number of letters in all the substitutions of any group is $n-\alpha$, n being the degree of the group and α the number of its transitive constituents.*

The last corollary may be proved as follows : Since G contains only $g \div n$ substitutions that do not involve a_α it must contain $g - g/n = \frac{n-1}{n} g$ that involve a_α. Hence all the g substitutions of G contain $n \times \frac{n-1}{n} g = (n-1)g$ letters.

From this corollary we may directly derive the following :

Corollary III. Every transitive group contains at least $n-1$ substitutions of the n^{th} degree.

Corollary IV. If the order of a transitive group exceeds its degree it must contain substitutions of a lower than the n^{th} degree and hence n conjugate subgroups $G_1, G_2, \ldots\ldots, G_n$ whose degree is at most $n-1$. These n subgroups need not all be distinct.

We may divide the primitive groups into two classes. (1) Those whose order is equal to their degree—the *regular* primitive groups—and (2) those whose order is b times their degree, where b is a positive integer larger than 1.

We proceed to consider the first one of these classes. Since the average number of letters in its substitutions is $n-1$ it must contain $n-1$ substitutions of the n^{th} degree, *i. e.* all its substitutions except unity are of the n^{th} degree.

If any one of these $n-1$ substitutions consists of two or more cycles all of these cycles will be of the same order, *i. e.* they will all contain the same number of letters, otherwise some power of this substitution would at the same time differ from identity and not contain all the letters of the group.

We proceed to prove the following

THEOREM : *Whenever a regular group contains a substitution (s) which contains more than one cycle it is non-primitive.*

Let $s = a_1 a_2 \ldots\ldots a_r, b_1 \ldots\ldots\ldots\ldots$ Some substitution of G (s_1) replaces a_1 by b_1. If we transform s with respect to s_1 we have

$$s_1^{-1} s\, s_1 = b_1 b_2 \ldots\ldots b_r \ldots\ldots$$

If we assume that

$$b_\alpha = a_\beta \qquad (\alpha, \beta \overline{\overline{<}} r)$$

we have as a consequence that s_1 replaces a_α by a_β. This is also done by $s^{\beta-\alpha}$. Since only one substitution of G can perform this operation we have as a second consequence of the given assumption

$$s_1 = s^{\beta-\alpha}.$$

The latter of these transforms the cycle $a_1 a_2 \ldots\ldots a_r$ into itself and the former does not, the given assumption is therefore untenable and the cycle of b's must be distinct from the cycle of a's.

If these a's and b's do not include all the letters of G there must be some

substitution of G (s_2) which replaces a_1 by some new letter c_1. We now derive the substitution

$$s_2^{-1} s\, s_2 = c_1 c_2 \ldots\ldots c_2 \ldots\ldots$$

We have already proved that these c's are all different from the a's. It remains to show that they do not include any b.

From $$c_a = b_\beta$$

it would follow that s_2 replaced a_a by b_β and therefore that

$$s_2 = s^{\beta - a} s_1 .$$

This is impossible since the second member replaces the a's by the b's and the first replaces a_1 by c_1.

Continuing in this manner we must finally exhaust the letters of G and obtain the l distinct cycles

$$a_1 a_2 \ldots\ldots a_r,\ b_1 b_2 \ldots\ldots b_r,\ \ldots\ldots,\ l_1 l_2 \ldots\ldots l_r$$

where $lr = n$, the degree of G.

We proceed to prove that these cycles may be used as systems of non-primitivity. This is, of course, included in the proof that the substitutions composed of these cycles

$$a_1 a_2 \ldots\ldots a_r.\ b_1 b_2 \ldots\ldots b_r \ \ldots\ldots\ l_1 l_2 \ldots\ldots l_r \equiv t$$

is transformed into itself by all the substitutions of G.

Let s_a represent any substitution of G; we desire to prove that

$$s_a^{-1} t s_a = t.$$

If s_a replaces c_γ by b_β we have

$$s_a = s_2^{-1} s^{\beta - \gamma} s_1 .$$

The second member replaces $c_{\gamma+\rho}$ by $b_{\beta+\rho}$ where ρ satisfies the congruence

$$\gamma + \rho,\ \beta + \rho \equiv \delta \pmod r,\ (\delta = 1, 2, \ldots\ldots r).$$

Hence s_a must replace the c's in order by the b's in order. Since similar remarks apply to all the cycles it follows that s_a which is any substitution of G transforms t into itself and our theorem is proved.

By starting with the different cycles of G which contain the same letter we obtain different systems of non-primitivity for the same group.*

*Cf. Jordan, Traite des Substitutions, §75; and Netto, Theory of Substitutions (American Edition), §98.

From the last theorem we see that a regular group cannot be primitive unless it is generated by a single cycle involving a prime number of letters. Since such a group must be primitive we have the following

THEOREM : *The regular primitive groups and the prime numbers have a* 1,1 *correspondence ; i. e. for each prime number there is one regular primitive group and for each regular primitive group there is one prime number.*

[To be Continued.]

THE CENTROID OF AREAS AND VOLUMES.

By G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

[Continued from February Number.]

II. VOLUMES. Let the density vary as $x^{h-1}y^{k-1}z^{l-1}$. Then

$$\bar{x}=\frac{\int\int\int x^hy^{k-1}z^{l-1}dxdydz}{\int\int\int x^{h-1}y^{k-1}z^{l-1}dxdydz},\quad \bar{y}=\frac{\int\int\int x^{h-1}y^kz^{l-1}dxdydz}{\int\int\int x^{h-1}y^{k-1}z^{l-1}dxdydz},$$

$$\bar{z}=\frac{\int\int\int x^{h-1}y^{k-1}z^ldxdydz}{\int\int\int x^{h-1}y^{k-1}z^{l-1}dxdydz}.$$

$$\therefore \bar{x}=\frac{\dfrac{a^{h+1}b^kc^l}{\dfrac{8}{(2m+1)(2n+1)(2p+1)}}\;\dfrac{\Gamma\left\{\frac{h+1}{2}(2m+1)\right\}\Gamma\left\{\frac{k}{2}(2n+1)\right\}\Gamma\left\{\frac{l}{2})2p+1)\right\}}{\Gamma\left\{\frac{h+1}{2}(2m+1)+\frac{k}{2}(2n+1)+\frac{l}{2}(2p+1)+1\right\}}}{\dfrac{a^hb^kc^l}{\dfrac{8}{(2m+1)(2n+1)(2p+1)}}\;\dfrac{\Gamma\left\{\frac{h}{2}(2m+1)\right\}\Gamma\left\{\frac{k}{2}(2n=1)\right\}\Gamma\left\{\frac{l}{2}(2p+1)\right\}}{\Gamma\left\{\frac{h}{3}(2m+1)+\frac{k}{2}(2n+1)+\frac{l}{2}(2p+1)+1\right\}}}$$

$$\therefore \bar{x} = \frac{\Gamma(hm+m+\frac{h+1}{2})\Gamma(hm+kn+lp+\frac{h+k+l}{2}+1)}{\Gamma(hm+\frac{h}{2})\Gamma(hm+kn+lp+m+\frac{h+k+l+1}{2}+1)}a \text{ ....} (C).$$

$$\bar{y} = \frac{\Gamma(kn+n+\frac{k+1}{2})\Gamma(hm+kn+lp+\frac{h+k+l}{2}+1)}{\Gamma(kn+\frac{k}{2})\Gamma(hm+kn+lp+n+\frac{h+k+l+1}{2}+1)}b \text{ ....} (D).$$

$$\bar{z} = \frac{\Gamma(lp+p+\frac{l+1}{2})\Gamma(hm+kn+lp+\frac{h+k+l}{2}+1)}{\Gamma(lp+\frac{l}{2})\Gamma(hm+kn+lp+p+\frac{h+k+l+1}{2}+1)}c \text{ ....} (E).$$

This gives the centroid of the eighth part of the volume whatever may be the values of h, k, l, m, n, p.

Let $m=n=p$, and also let the density vary as xyz so that $h=k=l=2$.

$$\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(3m+\frac{3}{2})\Gamma(6m+4)}{\Gamma(2m+1)\Gamma(7m+\frac{9}{2})}.$$

Let $m=0$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(\frac{3}{2})\Gamma(4)}{\Gamma(1)\Gamma(\frac{9}{2})} = \frac{16}{35}$, for $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$

Let $m=1$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(\frac{9}{2})\Gamma(10)}{\Gamma(3)\Gamma(\frac{23}{2})} = \frac{2^{13}}{11.13.17.19}$,

for $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1.$

Let $m=\frac{3}{2}$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(6)\Gamma(13)}{\Gamma(4)\Gamma(15)} = \frac{10}{91}$,

for $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1,$

the centroid of the volume bounded by the positive portion of the co-ordinate planes.

Let $m=n=p$, and let the density be the same throughout the solid so that, $h=k=l=1$

$$\therefore \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(2m+1)\Gamma(3m+\frac{5}{2})}{\Gamma(m+\frac{1}{2})\Gamma(4m+3)}.$$

Let $m=0$, $\therefore \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(1)\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(3)}=\frac{3}{8}$, for $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1.$

Let $m=1$, $\therefore \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(3)\Gamma(\frac{11}{2})}{\Gamma(\frac{3}{2})\Gamma(7)}=\frac{21}{128}$, for $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1.$

Let $m=\frac{3}{2}$, $\therefore \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(4)\Gamma(7)}{\Gamma(2)\Gamma(9)}=\frac{3}{28}$, for $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1.$

Let $m=n=p$, and let the density vary as xy, so that $h=k=2$, $l=1$,

$$\therefore \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(5m+\frac{7}{2})}{\Gamma(2m+1)\Gamma(6m+4)},\quad \frac{\bar{z}}{c}=\frac{\Gamma(2m+1)\Gamma(5m+\frac{7}{2})}{\Gamma(m+\frac{1}{2})\Gamma(6m+4)}.$$

Let $m=0$, $\therefore \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{7}{2})}{\Gamma(1)\Gamma(4)}=\frac{5\pi}{32}$,

$$\frac{\bar{z}}{c}=\frac{\Gamma(1)\Gamma(\frac{7}{2})}{\Gamma(\frac{1}{2})\Gamma(4)}=\frac{5}{16}, \text{ for} \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1.$$

Let $m=1$, $\therefore \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\Gamma(\frac{9}{2})\Gamma(\frac{17}{2})}{\Gamma(3)\Gamma(10)}=\frac{5.7.11.13.15\pi}{2^{20}}$,

$$\frac{\bar{z}}{c}=\frac{\Gamma(3)\Gamma(\frac{17}{2})}{\Gamma(\frac{3}{2})\Gamma(10)}=\frac{5.11.13}{2^{13}}, \text{ for} \left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1.$$

Let $m=\frac{3}{2}$,— $\therefore \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\Gamma(6)\Gamma(11)}{\Gamma(4)\Gamma(13)}=\frac{5}{33}$,

$$\frac{\bar{z}}{c}=\frac{\Gamma(4)\Gamma(11)}{\Gamma(2)\Gamma(13)}=\frac{1}{22}, \text{ for} \left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1.$$

Let $m=n=p$, and let the density vary as x so that $h=2$, $k=l=1$.

$$\therefore \frac{\bar{x}}{a} = \frac{\Gamma(3m+\frac{3}{2})\Gamma(4m+3)}{\Gamma(2m+1)\Gamma(5m+\frac{7}{2})}, \quad \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(2m+1)\Gamma(4m+3)}{\Gamma(m+\frac{1}{2})\Gamma(5m+\frac{7}{2})}.$$

Let $m=0$, then for $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$

$$\frac{\bar{x}}{a} = \frac{\Gamma(\frac{3}{2})\Gamma(3)}{\Gamma(1)\Gamma(\frac{7}{2})} = \frac{8}{15}, \quad \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(1)\Gamma(3)}{\Gamma(\frac{1}{2})\Gamma(\frac{7}{2})} = \frac{16}{15\pi}.$$

Let $m=1$, then for $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1$

$$\frac{\bar{x}}{a} = \frac{\Gamma(\frac{9}{2})\Gamma(7)}{\Gamma(3)\Gamma(\frac{17}{2})} = \frac{2^7}{3.11.13}, \quad \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(3)\Gamma(7)}{\Gamma(\frac{3}{2})\Gamma(\frac{17}{2})} = \frac{2^{14}}{5.7.9.11.13\pi}.$$

Let $m=\frac{3}{2}$, then for $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1$

$$\frac{\bar{x}}{a} = \frac{\Gamma(6)\Gamma(9)}{\Gamma(4)\Gamma(11)} = \frac{2}{9}, \quad \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(4)\Gamma(9)}{\Gamma(2)\Gamma(11)} = \frac{1}{15}.$$

Thus we could multiply examples almost without number.

If $a=b$ we get another series of areas.

If $a=b=c$ we get another series of solids.

If $b=c$ or $a=c$ we get still another series of solids.

But formulæ (A), (B), (C), (D), (E) apply to them all.

One more example and we will proceed to the discussion of surfaces. Let the density vary as x^3y^2z, and let the equation to the surface be

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{5}} + \left(\frac{z}{c}\right)^{\frac{2}{7}} = 1,$$

so that $h=4$, $k=3$, $l=2$, $m=1$, $n=2$, $p=3$

$$\therefore \bar{x} = \frac{\Gamma(\frac{15}{2})\Gamma(\frac{43}{2})}{\Gamma(6)\Gamma(23)}a = \frac{5.7.9.11.13.23.29.31.37.39.41\pi a}{2^{50}},$$

$$\bar{y} = \frac{\Gamma(10)\Gamma(\frac{43}{2})}{\Gamma(\frac{15}{2})\Gamma(24)}b = \frac{5.9.29.31.37.41b}{11.2^{26}},$$

$$\bar{z} = \frac{\Gamma(\frac{21}{2})\Gamma(\frac{43}{2})}{\Gamma(7)\Gamma(25)}c = \frac{5.7.9.11.13.13.17.19.29.31.37.41\pi c}{2^{57}}.$$

The prodigious amount of work to accomplish this by the ordinary method would be impossible.

[To be Continued.]

THE ANGLE-SUM ACCORDING TO PLAYFAIR.

By Professor JOHN N. LYLE, Ph. D., Westminster College, Fulton, Mo.

In Playfair's Euclid, pages 295 and 296, there is given a short method for finding the angle-sum of a rectilineal triangle.

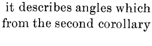
As the soundness of this method has been called in question by the Hyper-Space theorists, it is incumbent upon teachers of geometry to examine both the method itself and the criticisms to which it has been subjected.

John Playfair in treating of the angle-sum says—"It is of importance in explaining the Elements of Science, to connect truths by the shortest chain possible ; and till that is done, we can never consider them as being placed in their *natural order*.

The reasoning in the first of the following propositions is so simple, that it seems hardly susceptible of abbreviation, and it has the advantage of connecting immediately two truths so much alike, that one might conclude, even from the bare enunciations, that they are but different cases of the same general theorem, viz., That all the angles about a point, and all the exterior angles of any rectilineal figure, are constantly of the same magnitude, and equal to four right angles.

DEFINITION.

If, while one extremity of a straight line remains fixed at A, the line itself turns about that point from the position AB to the position AC, it is said to describe the angle BAC contained by the lines AB and AC.

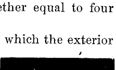


Corollary. If a line turn about a point from the position AC till it come into the position AC again, it describes angles which are together equal to four right angles. This is evident from the second corollary to the fifteenth, 1.

PROPOSITION I.

All the exterior angles of any rectilineal figure are together equal to four right angles.

1. Let the rectilineal figure be the triangle ABC, of which the exterior angles are DCA, FAB, GBC; these angles are together equal to four right angles.



Let the line CD, placed in the direction BC produced, turn about the point C till it coincide with CE, a part of the side CA, and have described the exterior angle DCE or DCA.

Let it then be carried along the line CA, till it be in the position AF, that is, in the direction of CA produced, and the point A remaining fixed, let it turn about A till it describe the angle FAB, and coincide with a part of the line AB.

Let it next be carried along AB until it come into the position BG, and by turning about B, let it describe the angle GBC so as to coincide with a part of BC.

Lastly, let it be carried along BC till it coincide with CD its first position.

Then, because the line CD has turned about one of its extremities till it has come into the position CD again, it has by the corollary to the above definition described angles which are together equal to four right angles ; but the angles which it has described are the three exterior angles of the triangle ABC, therefore the exterior angles of the triangle ABC are equal to four right angles.

2. If the rectilineal figure have any number of sides, the proposition is demonstrated just as in the case of a triangle. Therefore all the exterior angles of any rectilineal figure are together equal to four right angles.

Corollary 1. Hence, all the interior angles of any triangle are equal to two right angles. For all the angles of the triangle, both exterior and interior, are equal to six right angles, and the exterior being equal to four right angles, the interior are equal to two right angles."

In this demonstration of the angle sum Playfair evidently regards the method employed by him as legitimate, simple, direct and brief.

The Riemannian division of the Hyper-Space theorists assumes that a plane is the surface of an immense sphere, and that straight lines are curves that come back to their starting points, and, hence, raises the objection that such lines can not be slid along and then rotated as Playfair's demonstration requires.

This objection of the Riemannian School obviously rests on the false bottom that a plane is a spherical surface and that straight lines are curves. The foundation being insecure, that which is built thereon can not stand. The objection obliterates the distinction between spherical geometry and plane geometry.

If it be true that a plane is perfectly flat and that straight lines are devoid of curvature, the objection that we are considering is seen to have no force.

The Riemannian theorists tell us that for ought they know straight lines may be curves. They begin by doubting the truth of Euclid's second postulate—"That a terminated straight line may be produced to any length in a straight line"—and his Proposition XXXII, Book I. They are believers, also, as well as doubters. They believe that the angle-sum of a rectilineal triangle is greater than two right angles. They believe that if a straight line be extended it will ultimately return to the starting point.

The Euclidean geometers doubt these articles of Riemannian faith, and believe that the angle-sum of a rectilineal triangle is equal to two right angles, and that the longer a straight line is the further apart are its ends.

The Riemannians doubt what the Euclideans believe and believe what the Euclideans doubt. Those who undertake to teach both of these doctrines that contradict each other have failed to reckon with the logical laws of non-contradiction and excluded middle. We notice further that the Riemannian objection to Playfair's demonstration is in conflict with Proposition I of Lobatschewsky's

Theory of Parallels. Says the Russian Pangeometer—"A straight line fits upon itself in all its positions. By this I mean that during the revolution of the surface containing it the straight line does not change its place, if it goes through two unmoving points in the surface: (*i. e.*, if we turn the surface containing it about two points of the line, the line does not move)." These statements can not be made of any arc of any circle, and, hence, can not be made of Riemannian straight lines that are assumed to have constant positive curvature. What Lobatschewsky says respecting the straight line in his theorem I is inconsistent, of course, with his doctrine that the angle-sum is less than two right angles. But we are not quoting Lobatschewsky now to show that his theory is inconsistent with itself, but with that of Riemann.

Another objection to Playfair's demonstration is that a triangle drawn on a blackboard is not bounded by lines perfectly straight, since the surface of the board is uneven.

This objection does not hold against the triangle whose vertices are the three points A, B, and C in space and whose sides are destitute of curvature.

The geometer, whether he proceeds analytically or synthetically, naturally regards space as extending beyond himself on all sides without bounds, and between any two points A and B located therein he can draw an absolutely straight line with his mind, although he may be unable to do so with his hand. Some metaphysicians doubt these facts. What is it that they have not doubted? The function of a metaphysician, however, is to explain facts, not to doubt or discredit them.

Three points A, B, and C not in the same straight line may be located in trinally extended objective space and connected by the straight lines AB, AC, and BC. Hence, a rectilineal triangle in objective space is possible. When we say rectilineal triangle we do not mean a bogus triangle with wrinkled sides, but a genuine triangle with straight sides. When we say straight sides we do not mean wrinkled sides. The rectilineal triangle ABC of the geometer is perfect. His ability to cognize such a triangle is shown in the fact that he does cognize it. This fact, too, has been doubted. What a wonderful endowment that must be that enables man to people space with faultlessly perfect forms! This lofty power of intelligence in man, nay even his own doubt respecting it, differentiates him from the lower animals.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

34. Proposed by R. H. YOUNG, West Sunbury, Pennsylvania.

Prove (1) that $\frac{n(n+1)(2n+1)}{6}$ is a whole number for all values of n; and (2) prove that $\frac{n(n-1)(n+1)}{24}$ is a whole number when n is odd.

I. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

(1). $1^2+2^2+3^2+4^2+\ldots\ldots+n^2=\frac{n(n+1)(2n+1)}{6}=$ a whole number for all integral values of n.

(2). Let $n=2m+1=$ an odd number for all integral values of m.

$\therefore \frac{(n-1)n(n+1)}{24}=\frac{m(m+1)(2m+1)}{6}=$ same as (1).

II. Solution by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

As n and $n+1$ are consecutive numbers, one of them must be even and so divisible by two. But n must be of the form of $3p$, $3p+1$, or $3p+2$. If of the form of $3p$, it is divisible by three; if of the form $3p+2$, then $n+1$ or $3p+3$ is divisible by three; if of the form $3p+1$, then $(2n+1)$ becomes $6p+3$, and is divisible by three. Hence $n(n+1)(2n+1)$ is divisible by twice three, or six, whatever the value of n is.

2. $(n-1)n(n+1)$ of which the middle one is odd. One of every three consecutive numbers is always divisible by three: one of two consecutive *even* numbers is always divisible by four and the other by two. Hence $(n-1)n(n+1)$, when n is odd, is always divisible by $2\times3\times4$ or 24.

Also solved by *O. W. ANTHONY*, *M. A. GRUBER*, *EDGAR KESNER*, *E. W. MORRELL*, *J. SCHEFFER*, *E. L. SHERWOOD*, *B. F. YANNEY*, and *G. B. M. ZERR*.

35. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Decompose into the sum of two squares the number $13^2.61^3$.

I. Solution by E. L. SHERWOOD, A. M., Professor of Mathematics in Mississippi Normal College, Houston, Miss., and E. W. MORRELL, Department of Mathematics in Montpelier Seminary, Montpelier, Vermont.

$13^2.61^3=13^2.61^2.61=13^2.61^2(5^2+6^2)=13^2.61^2.5^2+13^2.61^2.6^2$.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Put $13^2.61^3=(p^2+q^2)^2(m^2+n^2)^3$, in which $p=3$, $q=2$, $m=6$, $n=5$. By decomposing into the sum of two squares, we find

$$(p^2+q^2)^2(m^2+n^2)^3=[m(p^2+q^2)(m^2-3n^2)]^2+[n(p^2+q^2)(3m^2-n^2)]^2=$$
$$[m(m^2+n^2)(p^2+q^2)]^2+[n(m^2+n^2)(p^2+q^2)]^2=$$
$$[m(p^2-q^2)(m^2-3n^2)\pm 2npq(3m^2-n^2)]^2+$$
$$[n(p^2-q^2)(3m^2-n^2)\mp 2mpq(m^2-3n^2)]^2=$$
$$[m(m^2+n^2)(p^2-q^2)\pm 2npq(m^2+n^2)]^2+[n(m^2+n^2)(p^2-q^2)\mp 2mpq(m^2+n^2)]^2,$$

making *six* sets of the sum of two squares.

Substituting the respective values of p, q, m, and n, we have

$$13^2.61^3=3042^2+5395^2=4758^2+3965^2=3810^2+4883^2$$
$$=6150^2+733^2=5490^2+2867^2=1830^2+5917^2.$$

Solved with these six sets of values by *A. H. BELL*, and with the five sets last in order by the *PROPOSER*. Also solved by *J. H. DRUMMOND*, *C. D. SCHMITT*, and *B. F. YANNEY*.

36. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the first six integral values of n in $\frac{n(n+1)}{2}=\square$.

I. Solution by Professor J. SCHEFFER, A. M., Hagerstown, Maryland, and O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

We have $n^2+n=2y^2$. Putting $n=\frac{t-1}{2}$, we obtain $t^2-8y^2=1$. Since $t=3$, $y=1$, satisfy this equation, we have $t=\frac{1}{2}[(3+2\sqrt{2})^m+(3-2\sqrt{2})^m]$, where for m successive integral numbers must be chosen. The required values of n we then obtain from the relation $n=\frac{t-1}{2}$.

For $m=1$, 2, 3, 4, 5, 6, in succession, we find in order the corresponding values of $t=3$, 17, 99, 577, 3363, 19601, and $n=1$, 8, 49, 288, 1681, 9800.

II. Solution by A. H. BELL, Hillsboro, Illinois.

Let $\frac{n(n+1)}{2}=\square=y^2$ say; then clearing of fractions, multiplying by 4, and adding 1 to both members, etc., $(2n+1)^2=8y^2+1=\square=x^2$ say.

$\therefore\ n=\frac{x-1}{2}$. Again $x^2-8y^2=1$. The 1st convergent for the $\sqrt{8}=\frac{3}{1}=\frac{x}{y}$ or the solution of this celebrated equation and the value of x and y can be found, on page 53, Vol. I. of MONTHLY.

The general value for x is $x_{n+1}=2x_1\times x_n-x_{n-1}$, hence $x_0=1$, $x_1=3$, $x_2=6\times3-1=17$, $x_3=6\times17-3=99$, $x_4=6\times99-17=577$, $x_5=6\times577-99=3363$, and $x_6=19601$, etc.

$\therefore$ The required values of $n=1$, 8, 49, 288, 1681, 9800, etc.

III. Solution by the PROPOSER.

When $\frac{n(n+1)}{2}=\square$, one of the factors, n and $n-1$, is a square and the

other two times a square. Being known *one* of the values of n in $\frac{n(n+1)}{2}=\square$, the value next succeeding as well as the value just preceding can be found by the following formula which I deduced by inspection:

$$\frac{n(n+1)}{2}=\left(2n_1+1\pm3\sqrt{\frac{n_1(n_1+1)}{2}}\right)^2$$

in which n_1 is a known value of n. By inspection we find that when $n=1$, $\frac{n(n+1)}{2}=\square=1^2$. Now put $n_1=1$, and substituting in the formula, we obtain $\frac{n(n+1)}{2}=6^2$ or 0^2, 6^2 being the $\square$ next succeeding and 0^2 the square just preceding 1^2. From $\frac{n(n+1)}{2}=6^2$, we obtain $n=8(=2\times2^2)$, or $-9(=-3^2)$, and $n+1=9(=3^2)$ or $-8(=-2\times2^2)$. Now put $n_1=8$, and substituting in the formula, we get $\frac{n(n+1)}{2}=(35)^2$ or $(-1)^2$, the positive value being the next succeeding square and the negative value the one just preceding, the latter being the square with which we started. From $\frac{n(n+1)}{2}=35^2$, we find $n=49$ or -50, and $n+1=50$, or -49. By continuing this process, we find the first six positive integral values of n in $\frac{n(n+1)}{2}=\square$, to be 1, 8, 49, 288, 1681, and 9800.

IV. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, O.

Let $n=p^2$ or p^2-1, since it must be a perfect power, or a perfect power less 1. Then $\frac{n(n+1)}{2}=\frac{p^2(p^2\pm1)}{2}=a^2$; whence, $p^2\pm1=\frac{2a^2}{p^2}=2q^2$ ......(1).

Adding $2q^2+4pq+p^2$ to each member of equation (1), we have,

$2q^2+4pq+2p^2\pm1=4q^2+4pq+p^2$; or $(2q+p)^2\pm1=2(q+p)^2$ ....(2).

Since equations (1) and (2) are the same in form, if we find one set of integral values for p and q in (1), we can then readily find succeeding values by (2). Now, for $p=1$, $q=1$. $\therefore$ Other values are: 3 and 2; 7 and 5; 17 and 12; 41 and 29; 99 and 70; and so on. Then by formula $\frac{n(n+1)}{2}=\frac{p^2(p^2\pm1)}{2}$, the first positive integral values of n are found to be 1, 8, 49, 288, 1691, 9800.

Also solved by *J. H. DRUMMOND*, *C. D. SCHMITT*, *H. C. WILKES*, *G. B. M. ZERR*, and the *PROPOSER*.

ERRATA. On page 368 of December issue, line 4, for "(10+2)" read (10^m+2); line 9, at end, for "B^2" read $B_1{}^2$; line 12, for "B^2" read $B_1{}^2$, and for

"$(B+1+A_1)$" read $(B+1-A_1)$; line 19, for "hypothenuse" read hypotenuse; line 22, leave out comma after 6; line 26, for "p, b, d," read p, d, b,; line 30, for "13, 14, 15," read 13, 15, 14; page 369, line 8, for "from" read for; line 25, for "the" read their; line 35, for "$a^{mn}+1$" read a^m+1; page 370, line 2, insert a comma before the sign of equality; and credit J. H. Drummond with a solution of No. 32.

NOTES, CRITICISMS, ETC., BY ARTEMUS MARTIN, LL. D.

On page 285 Mr. Adcock gives "An Equation for the sum of Squares equal a Square" which he says he has not seen published. I used the same method in the *Mathematical Magazine*, Vol. II., page 71, to find *three* square numbers whose sum is a square; and in a paper I had read at the last meeting of the American Association for the Advancement of Science I found in the same way *four* squares whose sum is a square. It is easily seen that the formula may be extended so as to find any number of squares whose sum is a square.

Note on Solutions of Problem 27, pp. 329-331.—In the *Mathematical Magazine*, Vol. II., No. 9, page 157, I have given a general method of finding any number (greater than two) of positive cube numbers whose sum is a cube, and on page 158 applied it to the case of five cubes, obtaining the set

$$6^3+11^3+13^3+18^3+20^3=26^3.$$

In Problem 42, p. 332, "$2a^2+2b^2-c^2+d^2$" should be $2a^2+2b^2=c^2+d^2$.

PROBLEMS.

45. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Penn.

Solve the equation $x^3+y^2=a^2$.

46. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

Give a general solution, finding such values of a and b in $x^2+x\sqrt{xy}=a$ and $y^2+y\sqrt{xy}=b$ as will make x and y integral.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

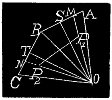
SOLUTIONS OF PROBLEMS.

27. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Penn.

Find the mean area of the *dodecagonal surface* formed by joining in order the points taken at random, one in each *sectoral triangle* of a regular inscribed dodecagon.

Solution by O. W. ANTHONY, Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Let AOB and BOC be two adjacent sectors of the regular dodecagon. Let the dodecagon be determined by its apothem $=a$.



Let $\angle P_1OB=\theta_1$, $\angle P_2OB=\theta_2$, $OP_1=\rho_1$, $OP_2=\rho_2$.

Then area of triangle $P_1OP_2=$

$$\tfrac{1}{2}\rho_1\rho_2\sin(\theta_1+\theta_2).$$

And average area of triangle $=$

$$\Delta=\frac{1}{2}\,\frac{\int_0^{\frac{1}{6}\pi}\int_0^{\frac{1}{6}\pi}\int_0^{OM}\int_0^{ON}\rho_1\rho_2\sin(\theta_1+\theta_2)d\rho_1d\rho_2d\theta_1d\theta_2}{\int_0^{\frac{1}{6}\pi}\int_0^{\frac{1}{6}\pi}\int_0^{OM}\int_0^{ON}d\rho_1d\rho_2d\theta_1d\theta_2}.$$

$$OM=a\sec(\theta_1-\frac{\pi}{12}).$$

$$ON=a\sec(\theta_2-\frac{\pi}{12}).$$

Then
$$\Delta=\frac{1}{8}a^2\frac{\int_0^{\frac{1}{6}\pi}\int_0^{\frac{1}{6}\pi}\sec^2(\theta_1-\frac{\pi}{12})\sec^2(\theta_2-\frac{\pi}{12})\sin(\theta_1+\theta_2)d\theta_1d\theta_2}{\int_0^{\frac{1}{6}\pi}\int_0^{\frac{1}{6}\pi}\sec(\theta_1-\frac{\pi}{12})\sec(\theta_2-\frac{\pi}{12})d\theta_1d\theta_2}.$$

The numerator may be written

$$\int_0^{\frac{1}{6}\pi}\left[\sec^2(\theta_2-\frac{\pi}{12})\int_0^{\frac{1}{6}\pi}\sec^2(\theta_1-\frac{\pi}{12})\sin[(\theta_1-\frac{\pi}{12})+(\theta_2+\frac{\pi}{12})]d\theta_1\right]d\theta_2.$$

The part under the last integral sign may be written, after expansion and some minor reductions,

$$\cos(\theta_2+\frac{\pi}{12})\int_0^{\frac{1}{6}\pi}\sec(\theta_1-\frac{\pi}{12})\tan(\theta_1-\frac{\pi}{12})d\theta_1+\sin(\theta_2+\frac{\pi}{12})\int_0^{\frac{1}{6}\pi}\frac{d\theta_1}{\cos(\theta_1-\frac{\pi}{12})}$$

$$=\cos(\theta_2+\frac{\pi}{12})\int_0^{\frac{1}{6}\pi}\sec(\theta_1-\frac{\pi}{12})+\sin(\theta_2+\frac{\pi}{12})\int_0^{\frac{1}{6}\pi}\log_e\frac{1+\tan\frac{1}{2}(\theta_1-\frac{\pi}{12})}{1-\tan\frac{1}{2}(\theta_1-\frac{\pi}{12})},$$

$$=0+2\sin(\theta_2+\frac{\pi}{12})\log_e\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}.$$

$\therefore$ The numerator may be written:

$$2\log_e\left(\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}\right)\int_0^{\frac{1}{6}\pi}\sec^2(\theta_2-\frac{\pi}{12})\sin(\theta_2+\frac{\pi}{12})d\theta_2.$$

The integral may be written

$$\int_0^{\frac{1}{6}\pi}\sec^2(\theta_2-\frac{\pi}{12})\left[\sin(\theta_2-\frac{\pi}{12})\cos(\theta_2-\frac{\pi}{12})\sin\frac{\pi}{6}\right]d\theta_2$$

=(after reductions similar to those above)

$$\tfrac{1}{2}\int_0^{\frac{1}{6}\pi}\frac{d\theta_2}{\cos(\theta_2-\frac{\pi}{12})}=\tfrac{1}{2}\log_e\left(\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}\right)$$

$\therefore$ The numerator reduces to

$$\left[\log_e\left(\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}\right)\right]^2.$$

It may also be shown that the denominator reduces to

$$\left[\log_e\left(\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}\right)\right]^2.$$

$\therefore \Delta=\frac{1}{8}a^2$.

And the mean area of dodecagon=area of 12 such triangles $=\frac{3}{2}a^2$.

Solutions of this problem were received from G. B. M. Zerr and the Proposer, the latter furnishing two solutions.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

30. Proposed by R. J. ADCOCK, Larchland, Warren County, Illinois.

When the sum of the distances of a point of a plane surface, from all other points, is a minimum, that point is the center of gravity of the plane surface.

IV. Discussion by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

I. Consider the following problem: Find a point within a plane surface such that the sum of the n^{th} power of the distances to all other points of the surface shall be a minimum.

$$S=\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^n dxdy.$$

For minimum—

$$\frac{dS}{dx_1}=2n\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1}(x_1-x)dxdy=0 \;\ldots\ldots\ldots\ldots(1),$$

$$\frac{dS}{dy_1}=2n\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1}(y_1-y)dxdy=0 \;\ldots\ldots\ldots\ldots(2).$$

(1) and (2) may be satisfied in several ways.

(A). The curve may be such that the integration in question performed over the surface reduce to zero.

(B). $\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1}dxdy=0$, or,

$$\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1}dxdy=C.$$

$$(C).\begin{cases}(x_1-x)dxdy=0, \text{ or} \int\int(x_1-x)dxdy=C_1\ldots\ldots\ldots\ldots\ldots\ldots(3),\\ (y_1-y)dxdy=0, \text{ and} \int\int(y_1-y)dxdy=C_2\ldots\ldots\ldots\ldots\ldots\ldots(4).\end{cases}$$

We shall only consider (C), as it is the only one which leads to the consideration of the center of gravity.

$$\text{From (3) and (4), } x_1 = \frac{C_1 + \int\int x dx dy}{\int dx dy}, \text{ and } y_1 = \frac{C_2 + \int\int x dx dy}{\int dx dy}.$$

Therefore (x_1, y_1) is the center of gravity only when C_1 and C_2 are zero. For this condition to be fulfilled the first member of (3) and (4) must be evidently zero. From (1) and (2) we see that this will be true generally only when $n-1=0$, *i. e.*, $n=1$. Hence there can be no *general* proposition except for the sum of the squares of the distances.

$$\text{II.} \quad u = \int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^n dx dy.$$

$$\frac{du}{dx_1} = 2n\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1}(x_1-x)dxdy = 0,$$

$$\frac{du}{dy_1} = 2n\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1}(y_1-y)dxdy = 0.$$

$$\text{Whence } x_1 = \frac{\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1} xdxdy}{\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1} dxdy},$$

$$\text{and } y_1 = \frac{\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1} ydxdy}{\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1} dxdy}.$$

For (x_1, y_1) to be identical with center of gravity $n-1$ must be zero.

III. Prof. Zerr in his proof has $S=\int DdA$. He then writes $\frac{dS}{dx_1} = \frac{(x-x_1)dA}{D}$. He should have written $\frac{dS}{dx_1} = \int\frac{x-x_1}{D}.\,dA$; the integration was with respect to A and the differentiation with respect to x_1, and the two do not destroy each other.

IV. The proposition fails to hold for the simplest case imaginable, an indefinitely narrow rectangle, or straight line. Thus let AB be a straight line, P any point on that line. $AP=a$, $AB=l$, $PQ=x$. Then the sum of distances from

A ——— P ——— Q ——— B

$P=S=\int_{-a}^{l-a} xdx=\frac{1}{2}[l^2-2al]$. $\frac{dS}{da}=-2l=0$ for minimum, *i. e.*, $l=0$ which is an absurdity. The sum of squares a minimum *will* hold in this case.

V. The same proof that Prof. Zerr gives will hold for *any* power of the distance, which proposition is highly improbable.

31. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Penn.

In order that a vertical cylindric stalk may be severed by a blow of minimum force, the stalk must be struck at what inclination by a sharp wedge-shaped blade?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Ark.-Tex.

Let $f\phi(\theta)=$the force necessary to sever a unit of area, where θ is the inclination to the horizon. Let $r=$radius of stalk.

$\therefore$ The area of section made in cutting is $\pi r^2\sec\theta$, the area of an ellipse with semi-axes r and $r\sec\theta$. $\therefore$ $\pi r^2 f\sec\theta\phi(\theta)=$a minimum. This can be made a minimum when $\phi(\theta)$ is known. If $f\phi(\theta)=a+b\cos^2\theta$, then $\theta=\sin^{-1}\sqrt{1-\frac{a}{b}}$.

32. Proposed by S. H. WRIGHT, M. D., A. M., Ph. D., Penn Yan, New York.

Intermittent reflections of flashes of light on a clear sky after dark, indicated a storm was progressing *below* the horizon. Refraction of 34′ on the horizon, brought the upper edge of the storm-cloud up to the horizon, and was just visible. How far off was the storm if the cloud was one mile above the earth?

I. Solution by the PROPOSER.

In the plane triangle ABC, let C be the center of the Earth, A the place of the observer, and B that of the cloud. Then $AC=$Earth's mean radius$=3959$ miles, $=b$, $BC=3960$ miles, $=a$, $AB=c$, the required distance. The angle $BAC=$the nadir distance of the cloud, being $90^\circ-34'=89^\circ26'=A$. Then

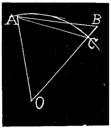
$$\sin B=\frac{b\sin A}{a}. \quad \therefore B=88^\circ35'36'', \text{ and } 180^\circ-(A+B=1^\circ58'24''=C, \text{ and}$$

$$c=\frac{b\sin C}{\sin B}=\frac{a\sin C}{\sin A}=136.367 \text{ miles.}$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Ark.-Tex.

Let A be the position of the observer, B the cloud, O the center of the earth, $R=$mean radius of the earth$=3958$ miles.

$$\therefore AC=2R\sin\tfrac{1}{2}AOC. \quad \angle ACB=\frac{\pi}{2}+\tfrac{1}{2}AOC, \quad \angle BAC=\tfrac{1}{2}AOC-34'.$$



$$\therefore AC = \frac{BC\cos(AOC-34')}{\sin(\frac{1}{2}AOC-34')} \;\ldots\ldots\;\ldots\ldots\ldots\ldots(1).$$

$$AB = \frac{BC\cos\frac{1}{2}AOC}{\sin(\frac{1}{2}AOC-34')} \;\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$BC=1$ from (1). $\quad 2R\sin\frac{1}{2}AOC = \dfrac{\cos(AOC-34')}{\sin(\frac{1}{2}AOC-34')}$.

$\therefore \cos(AOC-34') = \dfrac{(1+R)\cos 34'-1}{R+1}$. $\quad \therefore \; AOC = 1°58'16''$. From (2) $AB = 136.778$ miles.

ERRATA. On page 56, second line from top, for "$-2562Z^3$" read $-256Z^3$; sixth line from top for "$Z=2.750413$" read $Z=2.750458368$, and for "$WR=1.5248$" read $WR=1.2963390864$; and in Note, second line from bottom, for "$z=WR$" read $x=WR$.

PROBLEMS.

38. Proposed by S. H. WRIGHT, M. D., A. M., Ph. D., Penn Yan, New York.

In latitude $42°30'$ north $=\lambda$, at what angle with the horizon, will the sun rise, its declination $=22°$ north $=\delta$?

39. Proposed by SETH PRATT, C. E., Assyria, Michigan.

The pendulum of a clock which gains 6 seconds in 1 hour and 13 minutes, makes 6000 vibrations in 1 hour and $9\frac{1}{2}$ minutes. What is the length of the pendulum? And what length should it have to keep true time?

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

The Origin of π.—At least from the time of Archimedes π has stood for the number expressing how many times the diameter the circumference is. It is the initial letter of the Greek word περιφέρεια, meaning periphery. If the diameter is taken as a unit, then π stands for the periphery, or circumference. This is in reply to query of Lottie Smith in December Number.

BENJ. F. YANNEY,
Mount Union College, Alliance, Ohio.

An Expression for π.—Though the result is not new, I have not seen it developed as follows:

$$\text{Since } e^{i\pi}=-1, \quad \therefore\ i\pi\log e=\log(-1).$$

$$\therefore\ \pi=\frac{\log(-1)}{\sqrt{-1}}.$$

BENJ. F. YANNEY.

Referring to the Note of R. Greenwood in the December Number, I would state that (1) probably the other root was infinite. Thus the equations $x^2-y^2=5$ and $x+y=5$ have roots $x=3$ or ∞, and $y=2$ or $-\infty$. (2) The proof that imaginary roots enter in pairs assumes that all the coefficients are real. The equation $x^2-bix=a^2-abi$ has roots: a and $-a+bi$ but its coefficients are not all real. (3) The equation $\sqrt{2x_2-2}-(3x-5)=0$ or A must be multiplied by $\sqrt{2x^2-2}+(4x-5)=B$ or 0 to give a quadratic equation. The given equation is not of the second degree as Mr. Greenwood seems to imply but of the $\frac{2}{2}$ degree. An infinite number of equations can be written that have no roots at all: for instance, $2x-5+\sqrt{x^2-7}=0$ (or ?). This when combined with its congeneric $2x-5-\sqrt{x^2-7}=?$ or 0 gives a quadratic; the last expression takes both roots leaving no root for the first form. The demonstration that every equation has a root referred to equations free from surds.

H. C. WHITAKER,
Manual Training School, Philadelphia.

Another Reply. By squaring the equation, we get $2x^2-2=(3x-5)^2\ldots(1)$ $=2x^2-2-(3x-5)^2=0$, which is equivalent to transposing the member $3x-5$, and then multiplying the equation by $\sqrt{2x^2-2}+(3x-5)=0$. By doing this we have really introduced a new equation, which is satisfied for $x=\frac{9}{7}$.

Observe that (1) is satisfied for both $x=3$, and $x=\frac{9}{7}$, for it contains both the original equation and the one introduced by the questionable operation of squaring. Therefore, if the given equation means the positive root of $(2x^2-2)=3x-5$, then 3 is the only value of x that will satisfy it.

If $\pm\sqrt{2x^2-2}=3x-5$, both 3 and $\frac{9}{7}$ will.

BENJ. F. YANNEY,
Mount Union College, Alliance, Ohio.

Note on Solution IV., Page 190. It does not follow that triangles AEL and ADK are equal because the triangles AEL and ADK are similar respectively to AFN and AGM, and the solution fails.

I would like to see a direct proof of this problem. It is said that the mathematician Todhunter failed to produce a direct proof of it.

GEORGE LILLEY,
394 Hall Street, Portland, Oregon.

Problem by Euler. One answer is given, but he adds there are many more. Legendre asks for a general solution as Euler's solution is lost: and he says such

a solution would be very much prized by mathematicians, if it could be given.

1st. The sum of the squares of each, horizontal, vertical, or diagonal rows shall be equal,—10 conditions.

A	B	C	D
E	F	G	H
I	K	L	M
N	O	P	Q

68	-29	41	-37
-17	31	79	-32
59	28	23	61
-11	-77	8	49

=Euler's Numbers.

2nd. The sum of their products taken two and two=0, taking any two rows, horizontal, vertical, and diagonals,—12 conditions.

$$AE+BF+CG+DH=0=AD+FG+KL+NQ, \text{ etc.}$$

HILLSBORO, ILL., MATHEMATICAL CLUB.

Note on No. 4—Miscellaneous. In regard to No. 4 Miscellaneous, I had worked the problem with the assumption made by Prof. Hume, but rejected my solution, as on further thought I did not consider the assumption warranted. The constantly changing curvature carries with it a change in actual contact as well as in the amount ground off, which I have not been able to analyze. The assumption made would seem to apply if the stones were kept pressed together with such a force as would not yield, and would cause the particles to overlap for a constant distance. This also would require a constantly changing pressure or adjustment.

I should like to ask whether any one knows of a principle which will apply to the effect of friction in a case of this kind.

C. W. M. BLACK,
Wesleyan Academy, Wilbraham, Mass.

Query. Is a man who writes for publication in a Mathematical Magazine a "Note on Helmholtz's use of the terms 'Surface' and 'Space' as identical in meaning", properly to be considered sane?

Again when he asks "Does the 'immortal' Helmholtz in his Lectures on the—'Origin and Significance of Geometrical Axioms'—use the terms 'surface' and 'space' as identical in meaning?" since Helmholtz never delivered any lectures under this title, would it be sane to attempt to answer?

G. B. HALSTED.

The equation from Bell's Algebra, quoted by Mr. Greenwood, (MONTHLY, Vol., p. 372) is consistent if the radical be given the double sign. The equation should be

$$\pm\sqrt{2x^2-2}=3x-5.$$

The value $x=3$ belongs to the upper sign, $x=\frac{9}{7}$ to the lower.

WM. E. HEAL.

The answer to query (Monthly, Vol. II., p. 247) is not satisfactory. It is true "We *have* no method of finding the cube root by means of a compass" [and rule] but that does not prove the *impossibility* of a solution. What I wish, is a rigorous proof of the impossibility of expressing the roots of a cubic equation by a geometrical construction.

WM. E. HEAL.

Concerning the value of *factorial zero*, Chrystal says (Text Book of Algebra, Part II., page 4) "Strictly speaking 0! has no meaning. It is convenient, however, to use it, with the understanding that its value is 1 ; by so doing we avoid the exceptional treatment of initial terms in many series."

WM. E. HEAL.

IS THERE MORE THAN ONE ILLIMITABLE SPACE?

The Metageometers assume without proof that there are many varieties of space, differing in curvature, in the number of dimensions and in extent. Is their assumption axiomatic or does it need proof? Is it not really inconsistent with the hypothesis that space is everywhere and illimitable ?

The Metageometers concede that the space that contains our Universe may for aught they know to the contrary, be trinally extended, i. e., through any point of it, whatever, three straight lines may be drawn mutually at right angles to each other. Notwithstanding this concession, they assume that there are two varieties of space at least, the number of whose dimensions is less than three.

They call a surface a variety of space that has *two* dimensions, and a line a variety of space that has *one* dimension.

The Euclidian geometers locate all their lines and surfaces in the one, trinally extended, illimitable space. They do not regard these lines and surfaces as distinct varieties of space that may be classed under an n-fold species.

Some of the Metageometers call a line one dimensional space, and a surface two dimensional space, apparently with the expectation that this ambiguous use of the word space will somehow assist them in ascending from our tridimensional space to a hypothetical one of four dimensions, and from that to one of five dimensions, and so on. This is certainly a most hazardous enterprise that they have undertaken. They are attempting to scale the transcendental heights of Hyper-space with an analogical ladder constructed out of defective timber. The two bottom rounds—one dimensional space and two dimensional space—are unable to endure the strain put upon them. We do not mount to trinally extended space from surfaces, nor to surfaces from lines. But we start with trinally extended space and in it locate surfaces and lines.

Successful ascent cannot be made from tridimensional space to fourdimensional space.

1st.—Because no one knows or can know the direction from 3-fold space to 4-fold, even if the latter exists.

2nd.—Because no one knows or can know that 4-fold space exists for the reason that the fundamental laws of thought are violated in every effort of the mind to cognize it. Legitimate thinking cannot proceed in violation of logical law, but stultification may do so. The so-called "generalized space" of the Metageometers is believed to be the joint product of pseudo-generalization, pseudo-analogical reasoning, and pseudo-analytical interpretation.

JOHN N. LYLE.

BOOKS AND PERIODICALS.

Trigonometry for Schools and Colleges. By Frederick Anderegg, A. M., Professor of Mathematics, and Edward Drake Roe, Jr., A. M., Associate Professor of Mathematics in Oberlin College. 8vo. Cloth, 108 pp. Boston : Ginn & Co.

This little work is a decided improvement over most modern treatises on trigonometry. It treats the subject with clearness and accuracy and leads the student to an easy acquaintance with modern higher mathematics.' A number of new features are introduced. This is the first book we have yet seen in which it is shown that Plane Trigonometry is a special case of Spherical Trigonometry. Many other subjects of equal interest and importance are discussed. The authors deserve much credit for this original and unique work.

B. F. F.

An Elementary Treatise on Rigid Dynamics. By W. J. Loudon, B. A., Demonstrator in Physics in the University of Toronto. 8vo. Cloth, 236 pp. Price, $2.25. New York : Macmillan & Co.

This is a most excellent treatise on Rigid Dynamics. The subjects treated are made very clear and the student is still further aided in grasping those complex and difficult principles by very beautiful and accurate diagrams. Any student who has mastered the calculus can take up this work without any difficulty. At the close of each subject is a list of problems. The book closes with 306 problems all of which are very interesting to the student of dynamics. Some of these excellent problems will appear in future numbers of the MONTHLY.

B. F. F.

Notations de Logique Mathématique. Par G. Peano, Professeur d'Analyse infinitésimale à l'Université de Turin. Introduction au Formulaire de Mathématique Publie par la *Revista di Matematico*, Turin. Pamphlet, 52 pages.

A very interesting and valuable treatment of the notations of mathematical logic.

B. F. F.

Periodico di Mathematica. By L'Insegnamento Secondario. Pubblicato per cura di Aurelio Lugli, Professor di matematica nel R. Istituto tecnico di Roma.

The January-February number of this magazine contains a number of important papers and the solutions of 7 problems. B. F. F.

El Progreso Mathemático Periodico de Mathemáticas Puras y Aplicadas. Director D. Zoel G. de Galdeano, Catedrático de Geometria Analica en la Universidad de Zaragoza.

In this journal are published problems which are proposed by the best mathematicians in the world. The solutions are illustrated by beautiful diagrams. B. F. F.

Annals of Mathematics. Ormond Stone, Editor, Office of Publication, University of Virginia. Bi-monthly. Price, $2.00.

The September (1895) number contains the following articles: On the Improbability of Finding Shoals in the Open Sea by Sailing over the Geographical Positions in which they are Charted. By Mr. G. W. Littlehale. Note on the Congruence $2^{4n} \equiv (-)^n (2n)!/(n!)^2$, where $2n+1$ is a prime. By Prof. Frank Morley. Equations and Variables Associated with the Linear Differential Equation. By Dr. Geo. F. Metzler. The Calculus of Variations. By Dr. Harris Hancock. B. F. F.

March Monthly Magazine Number of The Outlook. Price, $1. per year in advance. The Outlook Company, 13 Astor Place, New York.

The illustrated monthly "Magazine Number of *The Outlook* for March has nearly fifty pages of reading matter, and more illustrations than any of the previous issues. Dr. R. L. Dickinson writes as an expert on hygienic and practical aspects of "Bicycling for Women," with cuts showing just what is right and wrong about women's riding; Edward Everett Hale tells of the "Higher Life of Boston;" there is a pleasant "Spectator" talk about picturesque New Orleans; Charleston of to-day is compared with its ante-bellum life in Mr. W. J. Abbot's "From Atlanta to the Sea;" Martin Luther is the subject of a fine article by professor Harnack, the great German theologian; and Mr. A R. Kimball has a readable article about Penzance and the Newlyn school of artists. All these articles are fully illustrated. Ian Maclaren's novel gains in interest and humor.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year. Single number, 10 cents. Irvington-on-the-Hudson, New York.

The General of the Army, the General commanding the U. S. Corps of Engineers, Vice-Pres. Webb of the New York Central, and John Jacob Astor, compose *The Cosmopolitan Magazine's* Board of Judges to decide the merits of the Horseless Carriages which will be entered in the May trials, for which the *The Cosmopolitan* offers $3000 in prizes. This committee is undoubtedly the most distinguished that has ever consented to act upon the occasion of the trial of a new and useful invention. The interest which these gentlemen have shown in accepting places upon the committee is indicative of the importance of the subject, and that the contest itself will be watched with marked interest on both sides of the Atlantic. Frank Stockton's new story, "Mrs. Cliff's Yacht," which begins in the April *Cosmopolitan*, promises to be one of the most interesting ever written by that fascinating story-teller. Readers of "The Adventures of Captain Horn" will find in "Mrs. Cliff's Yacht" something that they have been waiting for.